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# THE HYPERBOLIC FUNCTIONS

BY

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THIS IS TO CERTIFY THAT THE THESIS PREPARED UNDER MY SUPERVISION BY

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ENTITLED The Hyperbolic Functions.

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# Introduction

Like many other branches of higher mathematics, the hyperbolic trigonometric functions have within recent years received much attention both for their own sake and for their many important applications. So rapid has been their development that some authors are beginning to introduce them into elementary text books on Integral Calculus and Differential Equations. Few text books on Trigonometry treat this topic, although it is so closely allied to that of the circular functions. This is due no doubt to the briefness of the usual school course rather than to the inherent difficulties of the subject. In view of this fact the prime object of the present writer was to provide an account and explanation of the hyperbolic functions which will render them intelligibly useful to the students of pure and applied mathematics.

Moreover the investigations in this and the allied branches of mathematics are so numerous and cover so wide a range that the literature of

the subject is distributed through numerous books and journals. Hence, the second object of this thesis is to combine in a single volume the results of all accessible investigations heretofore made in this subject, in order that the future investigator along this line may have a convenient and concise synopsis, containing all desirable, elementary information. Perchance, in making use of this thesis he may receive some suggestions to inspire him to carry on further research along some line closely connected with the hyperbolic functions and thus enrich the theory of them. With these two objects in mind, this thesis has been written.

In planning the order of discussing the different topics the writer has attempted to treat first the most elementary subjects and later discuss those more difficult to understand. Further he was guided more or less by the requirements of logical sequence and systematic arrangement.

The thesis is divided into eight chapters and each of these into several sections differing in number and length according to the importance of the subject treated. The first chapter is concerned entirely with the matter of definition, the definition from the analytical standpoint being taken as the fundamental one. The geometric definition is introduced, since no treatise on the hyperbolic

functions is complete without it, and since it is from their relation to the hyperbola that they take their name. However, it is not made use of in the work, as all the properties can be explained much more easily from the analytical standpoint. The second chapter takes up the subject of relations between the functions and is quite similar to the corresponding discussion for the circular functions in our trigometries. The following chapter treats of the differentials and integrals of the various functions and inverse functions. In the fourth chapter the series for the various functions are developed, the method of determining the numerical values and the curves of the functions being treated incidentally. The fifth chapter takes up the relation existing between the hyperbolic and circular functions when the arguments are connected by the relation $v = gdu$. The sixth chapter treats of functions of imaginary and complex arguments, the importance of which lies in the fact that these functions can be expressed as pure imaginaries or complex functions in the form $a + ib$, through the introduction of the circular functions, and further that the hyperbolic functions have an imaginary period. In these six chapters

the most important points belonging to the theory of the hyperbolic functions are treated. The last two chapters contain a few examples of the uses which can be made of the hyperbolic functions in pure and applied mathematics respectively. The thesis is concluded by a short sketch of the development of these functions, and a short of list of the references used.

# Chapter I

## Definitions

§1. Analytical Definition. Let the exponential function $a^x$ be of such a nature that it is the sum of two functions P and Q, say, and its reciprical $\frac{1}{a^x}$ or $a^{-x}$ be the difference of the same two functions; that is

$$P+Q = a^x \quad \text{and} \quad P-Q = a^{-x}.$$

Solving for P and Q in terms of $a^x$ and $a^{-x}$ we have

$$P = \frac{a^x + a^{-x}}{2}, \quad Q = \frac{a^x - a^{-x}}{2}.$$

Let us call P the hyperbolic cosine of $x$ with respect to the base $a$ and Q the hyperbolic sine of $x$ with respect to the base $a$, or symbolically

$$\cosh(x,a) = P, \text{ and } \sinh(x,a) = Q$$

The particular case for which $a = e$ the base of the Napierian or natural system of logarithms gives:-

$$\cosh(x,e) = \frac{e^x + e^{-x}}{2} \quad \text{and} \quad \sinh(x,e) = \frac{e^x - e^{-x}}{2}$$

For convenience, as in the case of logarithms the $e$ is usually omitted in the symbols $\cosh(x,e)$ etc. and they are written $\cosh x$, $\sinh x$, etc.

Analogously ~~to~~ the circular functions we obtain the following:-

$$\tanh(x,a) = \frac{\sinh(x,a)}{\cosh(x,a)} = \frac{a^x - a^{-x}}{a^x + a^{-x}}$$

$$\coth(x,a) = \frac{\cosh(x,a)}{\sinh(x,a)} = \frac{a^x - a^{-x}}{a^x + a^{-x}}$$

$$\operatorname{sech}(x,a) = \frac{1}{\cosh(x,a)} = \frac{2}{a^x + a^{-x}}$$

$$\operatorname{cosech}(x,a) = \frac{1}{\sinh(x,a)} = \frac{2}{a^x - a^{-x}}$$

For the case $a = \varepsilon$

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{\varepsilon^x - \varepsilon^{-x}}{\varepsilon^x + \varepsilon^{-x}}; \quad \coth x = \frac{\varepsilon^x + \varepsilon^{-x}}{\varepsilon^x - \varepsilon^{-x}}$$

$$\operatorname{sech} x = \frac{2}{\varepsilon^x + \varepsilon^{-x}}; \quad \operatorname{cosech} x = \frac{2}{\varepsilon^x - \varepsilon^{-x}}$$

The hyperbolic functions with respect to the base $a$ may however just as well be expressed as functions to the base $\varepsilon$; for since $a^x = \varepsilon^{x \log a}$,

$$\cosh(x,a) = \frac{a^x + a^{-x}}{2} = \frac{\varepsilon^{x \log a} + \varepsilon^{-x \log a}}{2} = \cosh(x \log a)$$

and $$\sinh(x,a) = \frac{a^x - a^{-x}}{2} = \frac{\varepsilon^{x \log a} - \varepsilon^{-x \log a}}{2} = \sinh(x \log a)$$

Since, therefore, the general hyperbolic function to the base $a$ is easily expressed as a function to the base $\varepsilon$, we shall henceforth consider the term hyperbolic functions as designating functions referred to the base of the system of natural logarithms.

§2. Analogy and Relation to the Circular Functions. — As already stated in the introduction, the hyperbolic functions are in many points analogous to the circular functions. The values of the circular functions in exponential form are:

$$\sin x = \frac{\varepsilon^{xi} - \varepsilon^{-xi}}{2i} \qquad \cos x = \frac{\varepsilon^{xi} + \varepsilon^{-xi}}{2}$$

and so on for the others, where $i = \sqrt{-1}$. The following relations between the two sets of

functions may be deduced by comparison:-

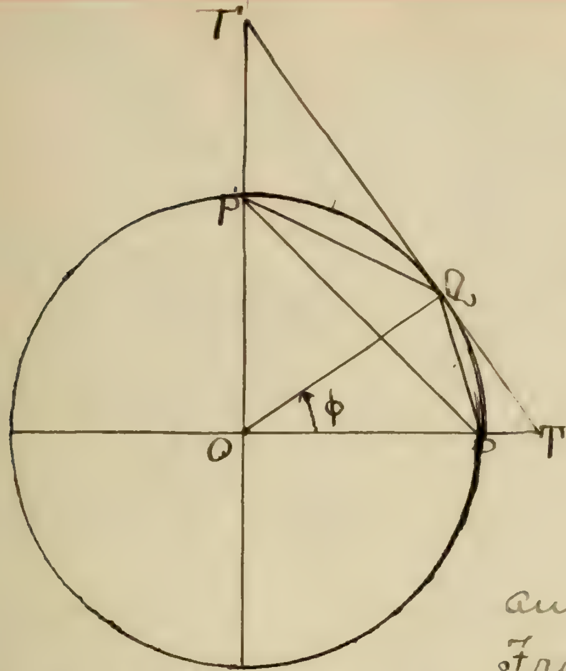
$$\cos xi = \frac{\varepsilon^{-x} + \varepsilon^{x}}{2} = \cosh x$$

$$\sin xi = \frac{\varepsilon^{-x} - \varepsilon^{x}}{2i} = i \sinh x$$

From this, it is apparent that if we have a circular function of an imaginary argument, it is easily expressed as a hyperbolic function of a real argument.

§3. Geometrical Analogy leading up to a Geometrical Definition of the Hyperbolic Functions. The analogy between the hyperbolic and circular functions is quite as apparent geometrically, since the hyperbolic functions bear the same relation to the hyperbola as the circular functions bear to the ellipse or, in their generalized form, to the ellipse. It may be stated here, that the hyperbolic functions bear no analogy whatever to the elliptic functions, which have grown out of attempts to rectify the ellipse, a problem quite foreign to the present one.

To lead up more easily to the geometrical definition of the hyperbolic functions we shall first define the circular functions in a way differing somewhat from that usually given in trigonometries but perfectly in harmony with them. With O as a center describe a circle of radius a. Let $\phi$ be the angle between two radii OP and OR and let OP'




be perpendicular to OP. Draw lines PP', PQ, and P'Q, and let TT' be tangent to the circle at Q. It can easily be shown that:

area sector $OPQ = \frac{1}{2}a^2\phi$,

area $\Delta\, OPQ = \frac{1}{2}a^2 \sin\phi$,

area $\Delta\, OP'Q = \frac{1}{2}a^2 \cos\phi$,

and area $\Delta\, OPP' = \frac{1}{2}a^2$

From this it follows that

$$\phi = \frac{\text{sector } OPQ}{\Delta\, OPP'}\,; \quad \sin\phi = \frac{\Delta\, OPQ}{\Delta\, OPP'}, \text{ and } \cos\phi = \frac{\Delta\, OP'Q}{\Delta\, OPP'}.$$

In a similar manner since

$\Delta\, OTQ = \frac{1}{2}a^2\tan\phi$; $\Delta\, OQT' = \frac{1}{2}a^2\cot\phi$,

$\Delta\, OTP' = \frac{1}{2}a^2\sec\phi$, and $\Delta\, OT'P = \frac{1}{2}a^2\operatorname{cosec}\phi$,

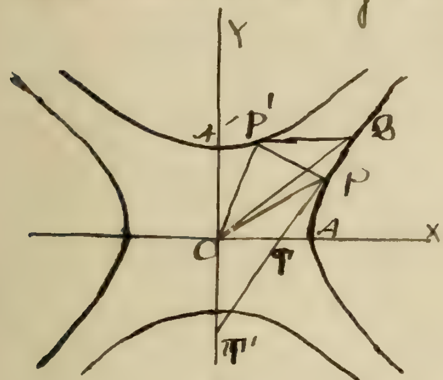
we get:-

$$\tan\phi = \frac{\Delta\, OTQ}{\Delta\, OPP'}, \quad \cot\phi = \frac{\Delta\, OQT'}{\Delta\, OPP'},$$

$$\sec\phi = \frac{\Delta\, OTP'}{\Delta\, OPP'}, \text{ and } \operatorname{cosec}\phi = \frac{\Delta\, OPT'}{\Delta\, OPP'}.$$

These formulae may be taken as definitions of the argument $\phi$ and its various functions, the basis of comparison being the area of the right triangle OPP' having two conjugate semi-diameters as sides. They can be immediately extended to the ellipse, the only modifications necessary being that OP and OP' shall be conjugate semidiameters. The area OPP' is then $\frac{1}{2}ab$ and $\phi$ is equal to the difference between the eccentric angles of P and Q

Passing to the hyperbola, since of two conjugate diameters only one meets the hyperbola in real points, the conjugate must be employed also. Let the diameter meet the hyperbola in P, and its conjugate, the conjugate hyperbola in P'. Also, draw line OQ, meeting the hyperbola in Q.




Then, we may define the hyperbolic functions in strict analogy to the circular functions defined above, as follows:-

$$u = \frac{\text{sector } OPQ}{\Delta OPP'}, \quad \sinh u = \frac{\Delta OPQ}{\Delta OPP'},$$

$$\cosh u = \frac{\Delta OQP'}{\Delta OPP'}, \quad \tanh u = \frac{\Delta OPT}{\Delta OPP'}$$

$$\operatorname{sech} u = \frac{\Delta OTP'}{\Delta OPP'}, \quad \operatorname{cosech} u = \frac{\Delta OT'P}{\Delta OPP'}, \quad \coth u = \frac{\Delta OT'Q}{\Delta OPP'}$$

Let us assume the hyperbola referred to its axes as axes of coordinates, its equation being

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Then, if $P = (x_1, y_1)$ and $Q = (x_2 y_2)$, it is easily shown

$$\sinh u = \frac{x_1 y_2 - x_2 y_1}{ab}; \quad \cosh u = \frac{x_1 x_2}{a^2} - \frac{y_1 y_2}{b^2}.$$

If now the axes are the conjugate diameters chosen for reference, the above formulae reduce to

$$\sinh u = \frac{\Delta OQA}{\Delta OAA'} = \frac{a y_2}{ab} = \frac{y_2}{b},$$

$$\cosh u = \frac{\Delta OA'Q}{\Delta OAA'} = \frac{\cancel{b}x_2}{ab} = \frac{x^2}{a}, \text{ etc.}$$

If $u_0 = \dfrac{\text{sector } OAP}{\Delta OAA'}$

$\sinh u_1 = \frac{y_1}{a}$, $\cosh u_1 = \frac{x_1}{a}$ etc.

If we make these definitions still less general by taking the hyperbola as rectangular, it is easily shown that if $(x_2 y_2)$ be the coordinates of $Q$

$$\sinh u = \frac{y_2}{a}, \quad \cosh u = \frac{x_2}{a}, \quad \tanh u = \frac{y_2}{x_2},$$
$$\operatorname{sech} u = \frac{a}{x_2}, \quad \operatorname{cosech} u = \frac{a}{y_2}, \quad \coth u = \frac{x_2}{y_2},$$

definitions quite similar to the ratio definitions of the circular functions.

§4 To show that these two sets of definitions are identical. Let $u_1 = \frac{\text{sector } OAP}{\Delta\, OAA'}$,

$$u_2 = \frac{\text{sector } OAQ}{\Delta\, OAA'_1}, \text{ and } u = \frac{\text{sector } OPQ}{\Delta OPP'}$$

Then it is evident, since $\Delta OAA' = \Delta OPP'$,

$$u = u_1 - u_2.$$

Now the area of the sector $OAP$ is easily shown by integral calculus to be $\frac{ab}{2}\log\left(\frac{x_1}{a} + \sqrt{\frac{x_1^2}{a^2} - 1}\right)$ and the area of the sector $OAQ$, $\frac{ab}{2}\log\left(\frac{x_2}{a} + \sqrt{\frac{x_2^2}{a^2} - 1}\right)$.

Hence $u_1 = \dfrac{\frac{ab}{2}\log\left(\frac{x_1}{a} + \sqrt{\frac{x_1^2}{a^2} - 1}\right)}{\frac{ab}{2}} = \log\left(\frac{x_1}{a} + \frac{y_1}{b}\right)$, since $(x_1, y_1)$ is on the hyperbola,

and $u_2 = \dfrac{\frac{ab}{2}\log\left(\frac{x_2}{a} + \sqrt{\frac{x_2^2}{a^2} - 1}\right)}{\frac{ab}{2}} = \log\left(\frac{x_2}{a} + \frac{y_2}{b}\right)$.

Hence $u = \log\left(\frac{x_2}{a} + \frac{y_2}{b}\right) - \log\left(\frac{x_1}{a} + \frac{y_1}{b}\right) = \log\left(\dfrac{\frac{x_2}{a} + \frac{y_2}{a}}{\frac{x_1}{a} + \frac{y_1}{b}}\right)$,

Or $\dfrac{\frac{x_2}{a} + \frac{y_2}{a}}{\frac{x_1}{a} + \frac{y_1}{b}} = e^{u}$.

Multiplying numerator and denominator by $\frac{x_1}{a} - \frac{y_1}{b}$, and remembering that the point $(x_1, y_1)$ is on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$,

$$\left(\frac{x_1}{a} - \frac{y_1}{b}\right)\left(\frac{x_2}{a} + \frac{y_2}{b}\right) = e^{u},$$

or $$\frac{x_1 x_2}{a^2} - \frac{y_1 y_2}{b^2} + \frac{x_1 y_2 - x_2 y_1}{ab} = e^{u}.$$

But by our definition in §3,

$$\frac{x_1 x_2}{a^2} - \frac{y_1 y_2}{b^2} = \cosh u \text{ and } \frac{x_1 y_2 - x_2 y_1}{ab} = \sinh u.$$

Hence $e^{u} = \sinh u + \cosh u.$

But this relation is also true by our analytical definition. Hence the two definitions are identical, and in what follows we can use either as a basis of proof for our theorems.

§5. The Inverse Hyperbolic Functions. Suppose

$x = \cosh u$, $y = \sinh u$ and so on.

Then in a manner similar to that used for the circular functions, we may write these equations

$\cosh^{-1} x = u$, $\sinh^{-1} y = u$, and so on,

read "the anti hyperbolic cosine of x", "the anti-hyperbolic sine of ~~x~~", and so on.

If $x = \cosh u$, $x = \dfrac{e^{u} + e^{-u}}{2}$ by §1.

Multiplying by $2e^{u}$ and transposing, we get.

$$e^{2u} - 2xe^{u} + 1 = 0.$$

Solving for $e^{u}$

$$e^{u} = x \pm \sqrt{x^2-1} = x + \sqrt{x^2-1} \text{ or } \frac{1}{x+\sqrt{x^2-1}}$$

Hence $u = \pm \log(x + \sqrt{x^2-1})$

The positive sign is always taken. Hence when $x$ is real $\cosh^{-1} x$ is a single valued function

In a way similar to the above we can show that

$$\sinh^{-1} x = \log(x + \sqrt{x^2+1}),$$

$$\tanh^{-1} x = \frac{1}{2} \log \frac{x-1}{x+1}.$$

# Chapter II.

## Elementary Relations between the Hyperbolic Functions of one Argument and of two or more Arguments.

§6. Negative Arguments. From our definition in §1 we have

$$\sinh u = \frac{e^u - e^{-u}}{2}, \quad \cosh u = \frac{e^u + e^{-u}}{2}.$$

If we substitute $-u$ for $u$ we see that

$$\sinh(-u) = \frac{e^{-u} - e^{u}}{2} = -\sinh u,$$

$$\cosh(-u) = \frac{e^{-u} + e^{u}}{2} = \cosh u.$$

Hence we have the following relations

$$\sinh(-u) = -\sinh u, \quad \cosh(-u) = \cosh u,$$

$$\tanh(-u) = -\tanh u, \quad \coth(-u) = -\coth u,$$

$$\operatorname{sech}(-u) = \operatorname{sech} u \quad \operatorname{cosech}(-u) = -\operatorname{cosech} u.$$

§7 Relations between different Functions of the same Argument. From our definition in §1

$$\cosh u + \sinh u = e^u,$$

$$\cosh u - \sinh u = e^{-u}.$$

Multiplying these two equations together we get

$$\cosh^2 u - \sinh^2 u = 1 \quad \ldots\ldots \quad (1)$$

This relation between the $\sinh u$ and $\cosh u$, together with the four elementary relations given in the definitions in §1, gives us five independent relations

between the six hyperbolic functions, such that each can be expressed in terms of the other five. By dividing this formula through by $\cosh^2 u$ we have ~~at once~~

$$1 - \tanh^2 u = \operatorname{sech}^2 u, \quad \dots\dots (2)$$

and dividing by $\sinh^2 u$,

$$\coth^2 u - 1 = \operatorname{cosech}^2 u, \quad \dots\dots (3)$$

direct relations between the hyperbolic secant and tangent, and cosecant and cotangent. The values of each function in terms of the other functions along with the proper signs are given in the following table:

| $\sinh u = s$ | $\cosh u = c$ | $\tanh u = t$ | $\operatorname{sech} u = x$ | $\operatorname{cosech} u = y$ | $\coth u = z$ |
|---|---|---|---|---|---|
| $s$ | $\sqrt{s^2+1}$ | $\frac{s}{\sqrt{s^2+1}}$ | $\frac{1}{\sqrt{s^2+1}}$ | $\frac{1}{s}$ | $\frac{\sqrt{s^2+1}}{s}$ |
| $\pm\sqrt{c^2-1}$ | $c$ | $\pm\frac{\sqrt{c^2-1}}{c}$ | $\frac{1}{c}$ | $\pm\frac{1}{\sqrt{c^2-1}}$ | $\pm\frac{c}{\sqrt{c^2-1}}$ |
| $\frac{t}{\sqrt{1-t^2}}$ | $\frac{1}{\sqrt{1-t^2}}$ | $t$ | $\sqrt{1-t^2}$ | $\frac{\sqrt{1-t^2}}{t}$ | $\frac{1}{t}$ |
| $\frac{1}{y}$ | $\pm\frac{\sqrt{1+y^2}}{y}$ | $\pm\frac{1}{\sqrt{1+y^2}}$ | $\pm\frac{y}{\sqrt{1+y^2}}$ | $y$ | $\pm\sqrt{1+y^2}$ |
| $\pm\frac{1}{\sqrt{z^2-1}}$ | $\pm\frac{z}{\sqrt{z^2-1}}$ | $\frac{1}{z}$ | $\pm\frac{\sqrt{z^2-1}}{z}$ | $\pm\sqrt{z^2-1}$ | $z$ |

§8. Addition Formulae. Since by §1

$e^u = \cosh u + \sinh u$ and $e^v = \cosh v + \sinh v$,

$$e^{u+v} = (\cosh u + \sinh u)(\cosh v + \sinh v)$$
$$= \cosh u \cosh v + \cosh u \sinh v + \sinh u \cosh v + \sinh u \cosh u$$

Similarly since

$e^{-u} = \cosh u - \sinh u$ and $e^{-v} = \cosh v - \sinh v$,

$$e^{-u-v} = (\cosh u - \sinh u)(\cosh v - \sinh v)$$

$$= \cosh u \cosh v - \cosh u \sinh v - \sinh u \cosh v + \sinh u \sinh v.$$

Now $\sinh(u+v) = \dfrac{e^{u+v} - e^{-u-v}}{2}$

and $\cosh(u+v) = \dfrac{e^{u+2v} + e^{-u-v}}{2}$.

Hence substituting the above values for $e^{u+v}$ and $e^{-u-v}$ we get

$$\sinh(u+v) = \sinh u \cosh v + \cosh u \sinh v \quad \ldots (1)$$

and $$\cosh(u+v) = \cosh u \cosh v + \sinh u \sinh v. \quad \ldots (2)$$

These formulae have been proved independently of whether u and v are positive or negative. They therefore hold just as well if we replace $v$ by $-v$. Remembering that $\cosh(-v) = \cosh v$ and $\sinh(-v) = -\sinh v$, by §6, we obtain the following:

$$\sinh(u-v) = \sinh u \cosh v - \cosh u \sinh v. \quad \ldots (3)$$

$$\cosh(u-v) = \cosh u \cosh v - \sinh u \sinh v \quad \ldots (4)$$

We may also let $u = v$ in (1) and (2) and we get functions ~~of~~ twice an argument in terms of functions of the argument, viz:-

$$\sinh 2u = 2 \sinh u \cosh u \quad \ldots\ldots (5)$$

$$\cosh 2u = \cosh^2 u + \sinh^2 u$$

$$= 1 + 2\sinh^2 u = 2\cosh^2 u - 1. \quad \ldots (6)$$

If ~~we~~ let $v = 2u$ we have

$$\sinh 3u = \sinh u \cosh 2u + \cosh u \sinh 2u$$

$$= \sinh u (1 + 2\sinh^2 u) + 2\cosh^2 u \sinh u$$

$$= \sinh u + 2\sinh^3 u + 2\sinh u (1 + \sinh^2 u)$$

$$= 3\sinh u + 4 \sinh^3 u \quad \ldots (7)$$

and

$$\cosh 3u = \cosh u \cosh 2u + \sinh u \sinh 2u$$
$$= \cosh u(2\cosh^2 u - 1) + 2\sinh^2 u \cosh u$$
$$= 4\cosh^3 u - 3\cosh u \quad \ldots\ldots (8)$$

Since $\tanh x = \dfrac{\sinh x}{\cosh x}$,

$$\tanh(u+v) = \frac{\sinh(u+v)}{\cosh(u+v)}$$
$$= \frac{\sinh u \cosh v + \cosh u \sinh v}{\cosh u \cosh v + \sinh u \sinh v}$$

Dividing numerator and denominator by $\cosh u \cosh v$ we get

$$\tanh(u+v) = \frac{\dfrac{\sinh u \cosh v + \cosh u \sinh v}{\cosh u \cosh v}}{\dfrac{\cosh u \cosh v + \sinh u \sinh v}{\cosh u \cosh v}}$$
$$= \frac{\tanh u + \tanh v}{1 + \tanh u \tanh v} \quad \ldots\ldots 9)$$

Replacing $v$ by $-v$ and remembering that $\tanh(-v) = -\tanh v$,

$$\tanh(u-v) = \frac{\tanh u - \tanh v}{1 - \tanh u \tanh v} \quad \ldots (10)$$

Also if $u = v$ in (8)

$$\tanh 2u = \frac{2\tanh u}{1 + \tanh^2 u} \quad \ldots\ldots (11)$$

Similar formulae can easily be derived for the hyperbolic secant, cosecant and cotangent either indepently or directly from the above relations

§9. Conversion Formulae. Adding and subtracting formulae (1) and (3) and (2) and (4) we get

$$\sinh(u+v) + \sinh(u-v) = 2\sinh u \cosh v$$
$$\sinh(u+v) - \sinh(u-v) = 2\cosh u \sinh v$$
$$\cosh(u+v) + \cosh(u-v) = 2\cosh u \cosh v$$
$$\cosh(u+v) - \cosh(u-v) = 2\sinh u \sinh v$$

Now letting $u+v=x$ and $u-v=y$ and hence $u=\frac{x+y}{2}$ and $v=\frac{x-y}{2}$, we have

$$\sinh x+\sinh y = 2\sinh\frac{x+y}{2}\cosh\frac{x-y}{2}$$
$$\sinh x-\sinh y = 2\cosh\frac{x+y}{2}\sinh\frac{x-y}{2}$$
$$\cosh x+\cosh y = 2\cosh\frac{x+y}{2}\cosh\frac{x-y}{2}$$
$$\cosh x-\cosh y = 2\sinh\frac{x+y}{2}\sinh\frac{x-y}{2}$$

No doubt the reader has noticed throughout this chapter the similarity between the relations between the hyperbolic functions and the corresponding relations of the circular functions. As a matter of fact, all these formulae could have been derived from those of the circular functions through the relations

$$\sin xi = i\sinh x$$
and
$$\cos xi = \cosh x$$

It may be stated further, that the fundamental relations as (1) of §7 and (1) & (2) of §8 could just as well have been derived from the geometrical definition given in §3.

# Chapter III

## Differentiation and Integration of the Hyperbolic Functions

§10. Derivatives of the Hyperbolic Functions. As in the case of the circular functions, the derivatives of the hyperbolic functions are easily found. They can be derived independently by the method of increments, and also by making use of their analytical definitions. Of these two methods we shall use the latter, although the other would have been just as good.

Since $\sinh u = \frac{e^u - e^{-u}}{2}$ and $\cosh u = \frac{e^u + e^{-u}}{2}$

$$\frac{d}{du}(\sinh u) = \frac{d}{du}\left(\frac{e^u - e^{-u}}{2}\right) = \frac{e^u + e^{-u}}{2} = \cosh u \quad (1)$$

and

$$\frac{d}{du}(\cosh u) = \frac{d}{du}\left(\frac{e^u + e^{-u}}{2}\right) = \frac{e^u - e^{-u}}{2} = \sinh u \quad (2).$$

These two derivatives are the fundamental ones and are sufficient to give us the derivatives of the other functions by ordinary differentiation

That is.
$$\frac{d}{du}(\tanh u) = \frac{d}{du}\left(\frac{\sinh u}{\cosh u}\right)$$
$$= \frac{\cosh^2 u - \sinh^2 u}{\cosh^2 u} = \frac{1}{\cosh^2 u}$$
$$= \operatorname{sech}^2 u \quad \ldots\ldots (3)$$

and $\frac{d}{du}\coth u = \frac{d}{du}\left(\frac{\cosh u}{\sinh u}\right)$

$= \frac{\sinh^2 u - \cosh^2 u}{\sinh^2 u} = -\frac{1}{\sinh^2 u}$

$= \operatorname{cosech}^2 u$ . . . . . (4)

Similarly $\frac{d}{du}(\operatorname{sech} u) = \frac{d}{du}\left(\frac{1}{\cosh u}\right)$

$= -\frac{\sinh u}{\cosh^2 u}$

$= -\operatorname{sech} u \tanh u$ . . . (5)

and $\frac{d}{du}(\operatorname{cosech} u) = \frac{d}{du}\left(\frac{1}{\sinh u}\right)$

$= -\frac{\cosh u}{\sinh^2 u}$

$= -\operatorname{cosech} u \coth u$ . . . (6)

It is evident from these formulae, that the hyperbolic sine and cosine reproduce themselves in their successive derivatives in a way, similar to the circular sine and cosine, except that the signs do not change. This gives a ready solution to the problem; "What function repeats itself in its second derivative?"

The similarity of the derivative forms of the circular functions to the corresponding hyperbolic is obvious.

§11. Differentials of the Inverse Hyperbolic Functions. Let $x = \sinh^{-1} u$.

Then $\sinh x = u$.

Differentiating $\cosh x\, dx = du$,

or since $\cosh x = \sqrt{1+\sinh^2 x}$,

$\sqrt{1+\sinh^2 x}\, dx = du$.

Hence $dx = d(\sinh^{-1} u) = \frac{du}{\sqrt{1+\sinh^2 x}} = \frac{du}{\sqrt{1+u^2}}$ . . (1)

Again let $x = \cosh^{-1}u$ or $\cosh x = u$

Differentiating $\sinh x\,dx = du$,

$$\sqrt{\cosh^2 x - 1}\,dx = du,$$

$$dx = d(\cosh^{-1}u) = \frac{du}{\sqrt{u^2-1}} \quad \ldots (2)$$

Similarly if $x = \tanh^{-1}u$,

$$du = \operatorname{sech}^2 x\,dx,$$

and $$dx = d(\tanh^{-1}u) = \frac{du}{1-u^2} \quad \ldots (3)$$

and in like manner

$$d(\coth^{-1}u) = \frac{-du}{u^2-1} \quad \ldots (4)$$

Of these two formulas [(3) + (4)] the first holds only when $u$ is less than one, and the second when $u$ is greater than one. (The reason for this will be explained in §15.)

Again, let $x = \operatorname{sech}^{-1}u$.

Then $$du = -\operatorname{sech} x \tanh x\,dx$$

$$= -\operatorname{sech} x\sqrt{1-\operatorname{sech}^2 x}\,dx,$$

and hence $$dx = -\frac{du}{u\sqrt{1-u^2}} \quad \ldots (5)$$

Similarly $$d(\operatorname{cosech}^{-1}u) = -\frac{du}{u\sqrt{u^2+1}} \quad \ldots (6)$$

§12 Integration of the Hyperbolic Functions

The following formulas follow immediately from §10 from the fact that integration is the inverse of differentiation.

$$\int \sinh u\,du = \cosh u \quad \ldots (1)$$

$$\int \cosh u\,du = \sinh u \quad \ldots (2)$$

$$\int \operatorname{sech}^2 u\,du = \tanh u \quad \ldots (3)$$

$$\int \operatorname{cosech}^2 u\,du = -\coth u \quad \ldots (4)$$

$$\int \operatorname{sech} u \tanh u\,du = -\operatorname{sech} u \quad \ldots\ldots (5)$$
$$\int \operatorname{cosech} u \coth u\,du = -\operatorname{cosech} u \quad \ldots\ldots (6)$$

Other formulae are easily derived as follows:

$$\int \tanh u\,du = \int \frac{\sinh u}{\cosh u}\,du$$
$$= \log \cosh u \quad \ldots\ldots (7)$$
$$\int \coth u\,du = \int \frac{\cosh u}{\sinh u}\,du = \log \sinh u \ldots (8)$$
$$\int \operatorname{sech} u\,du = \int \frac{du}{\cosh u} = \int \frac{\cosh u\,du}{\cosh^2 u}$$
$$= \int \frac{\cosh u\,du}{1+\sinh^2 u} = \tan^{-1} \sinh u^{*} \quad (9)$$
$$\int \operatorname{cosech} u\,du = \int \operatorname{cosech} u \frac{\operatorname{cosech} u - \coth u}{\operatorname{cosech} u - \coth u}\,du$$
$$= \int \frac{-\coth u \operatorname{cosech} u + \operatorname{cosech}^2 u}{\operatorname{cosech} u - \coth u}\,du$$
$$= \log(\operatorname{cosech} u - \coth u)$$
$$= \log \frac{1-\cosh u}{\sinh u} = \log \frac{2\sinh^2 \frac{u}{2}}{2\sinh\frac{u}{2}\cosh\frac{u}{2}}$$
$$= \log \tanh \frac{u}{2} \quad \ldots\ldots (10)$$

§13. Integration of Expressions resulting in the Inverse Hyperbolic Functions. We easily obtain the following integrals from the differentiation formulae of §11:-

$$\int \frac{du}{\sqrt{1+u^2}} = \sinh^{-1} u \qquad \int \frac{du}{\sqrt{a^2+u^2}} = \sinh^{-1}\frac{u}{a} \quad (1)$$
$$\int \frac{du}{\sqrt{u^2-1}} = \cosh^{-1} u \qquad \int \frac{du}{\sqrt{u^2-a^2}} = \cosh^{-1}\frac{u}{a} \quad (2)$$
$$\left[\int \frac{du}{1-u^2}\right]_{u<1} = \tanh^{-1} u \qquad \left[\int \frac{du}{a^2-u^2}\right]_{u<a} = \frac{1}{a}\tanh^{-1}\frac{u}{a} \quad (3)$$
$$\left[\int \frac{-du}{u^2-1}\right]_{u>1} = \coth^{-1} u \qquad \left[\int \frac{-du}{u^2-a^2}\right]_{u>a} = \frac{1}{a}\coth^{-1}\frac{u}{a} \quad (4)$$

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*Compare this result with that in §20.

$$\int\frac{-du}{u\sqrt{1-u^2}}=\operatorname{sech}^{-1}u \qquad \int\frac{-du}{u\sqrt{a^2-u^2}}=\frac{1}{a}\operatorname{sech}^{-1}\frac{u}{a}$$

$$\int\frac{-du}{u\sqrt{u^2+1}}=\operatorname{cosech}^{-1}u \qquad \int\frac{-du}{u\sqrt{u^2+a^2}}=\frac{1}{a}\operatorname{cosech}^{-1}\frac{u}{a}.$$

From these fundamental integrals the following are easily derived:

$$\int\frac{dx}{\sqrt{ax^2+2bx+c}}=\frac{1}{\sqrt{a}}\sinh^{-1}\frac{ax+b}{\sqrt{ac-b^2}},\ a \text{ positive } ac>b^2,$$

$$=\frac{1}{\sqrt{a}}\cosh^{-1}\frac{ax+b}{\sqrt{b^2-ac}},\ a \text{ positive } ac<b^2,$$

$$=\frac{1}{\sqrt{a}}\cos^{-1}\frac{ax+b}{\sqrt{b^2-ac}},\ a \text{ negative}.$$

$$\int\frac{dx}{ax^2+2bx+c}=\frac{1}{\sqrt{ac-b^2}}\tan^{-1}\frac{ax+b}{\sqrt{ac-b^2}},\ ac>b^2.$$

$$=\frac{1}{\sqrt{b^2-ac}}\tanh^{-1}\frac{ax+b}{\sqrt{ac-b^2}},\ \begin{cases} ax+b<\sqrt{b^2-ac}, \\ ac<b^2 \end{cases}$$

$$=-\frac{1}{\sqrt{b^2-ac}}\coth^{-1}\frac{ax+b}{\sqrt{b^2-ac}},\ \begin{cases} ac<b^2, \\ ax+b>\sqrt{b^2-ac} \end{cases}$$

$$\int\sqrt{x^2-a^2}\,dx=\frac{x}{2}\sqrt{x^2-a^2}-\frac{a^2}{2}\cosh^{-1}\frac{x}{a}$$

$$\int\sqrt{x^2+a^2}\,dx=\frac{x}{2}\sqrt{x^2+a^2}+\frac{a^2}{2}\sinh^{-1}\frac{x}{a}$$

$$\int\sqrt{a^2-x^2}\,dx=\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}$$

# Chapter IV

## Series for the Hyperbolic Functions; Variations of the Functions with "$u$"; Graphs of the Functions.

### §14. Series for the Hyperbolic Functions

From §1, by definition, $\sinh u = \frac{e^u - e^{-u}}{2}$.

But $$e^u = 1 + u + \frac{u^2}{\lfloor 2} + \frac{u^3}{\lfloor 3} + \frac{u^4}{\lfloor 4} + \dots$$

and $$e^{-u} = 1 - u + \frac{u^2}{\lfloor 2} + \frac{u^3}{\lfloor 3} + \frac{u^4}{4} - \dots$$

Hence by substitution

$$\therefore \sinh u = u + \frac{u^3}{\lfloor 3} + \frac{u^5}{\lfloor 5} + \dots \qquad (1)$$

Similarly, since $\cosh u = \frac{e^u + e^{-u}}{2}$,

$$\cosh u = 1 + \frac{u^2}{\lfloor 2} + \frac{u^4}{\lfloor 4} + \frac{u^6}{\lfloor 6} + \dots \qquad (2)$$

We can also develope these formulae from McLaurin's Theorem :-

$$f(u) = f(0) + u f'(0) + \frac{u^2}{\lfloor 2} f''(0) + \frac{u^3}{\lfloor 3} f'''(0) + \dots$$

If $f(u) = \sinh u$, $f'(u) = \cosh u$, $f''(u) = \sinh u$, ...

Then $f(0) = 0$, $f'(0) = 1$, $f''(0) = 0$, * ...

Hence $\sinh u = u + \frac{u^3}{\lfloor 3} + \frac{u^5}{\lfloor 5} + \frac{u^7}{\lfloor 7} + \dots$

Similarly or by differentiation

$$\cosh u = 1 + \frac{u^2}{\lfloor 2} + \frac{u^4}{\lfloor 4} + \frac{u^6}{\lfloor 6} + \dots$$

From the addition of these two we get

$$\cosh u + \sinh u = e^u$$

* These values may be obtained from substitution in the formulae $\frac{e^u - e^{-u}}{2}$, &c.

This last result would seem to be the result of reasoning in a circle, at least on the surface. If however, we obtain the differentiation formulae independently of the analytical definition, as it is possible to do, and obtain the values for sinh 0, cosh 0, from the geometrical definition, the result shows again the agreement of the two definitions.

From series (1) and (2) the following may be obtained by division:-

$$\tanh u = \frac{\sinh u}{\cosh u} = u - \frac{1}{3}u^3 + \frac{2}{15}u^5 + \frac{17}{315}u^7 + \dots \quad (3)$$

$$\operatorname{sech} u = \frac{1}{\cosh u} = 1 - \frac{1}{2}u^2 + \frac{5}{24}u^4 - \frac{61}{720}u^6 + \dots \quad (4)$$

$$u\coth u = u\frac{\cosh u}{\sinh u} = 1 + \frac{1}{3}u^2 - \frac{1}{45}u^4 + \frac{2}{945}u^6 + \dots \quad (5)$$

$$u\operatorname{cosech} u = u\frac{1}{\sinh u} = 1 \cancel{+} \frac{1}{6}u^2 + \frac{7}{360}u^4 - \frac{31}{15120}u^6 + \dots \quad (6)$$

These series are however seldom used, because there is no known law by which the coefficients progress. Moreover the tanh u, sech u, coth u, and cosech u can be easily found if the values of cosh u and sinh u are known.

§15. Variations of the Hyperbolic Functions

The above series give us a set of criteria by which we can tell the values of the different hyperbolic functions as u varies.

If u is zero it is evident that

$\cosh 0 = 1$, $\sinh 0 = 0$, $\tanh 0 = 0$,

$\operatorname{sech} 0 = 1$, $\operatorname{cosech} 0 = \infty$, $\coth 0 = \infty$.

If $u$ varies from zero to infinity, the values of $\cosh u$ and $\sinh u$ will increase indefinitely and become infinite when $u$ is infinite. It is apparent from the series that they approach infinity much faster than $u$, showing that $\cosh u$ and $\sinh u$ are infinites of a higher order than $u$. Although the hyperbolic sine and cosine approach infinity simultaneously they are never equal to each other while finite, as is evident from § 7, (1), viz. $\cosh^2 u - \sinh^2 u = 1$, $\cosh u$, therefore always being the greater.

Tanh $u$ will also increase from zero upward as $u$ increases, but it will never become greater than unity as is evident from the fact that $\tanh u = \frac{\sinh u}{\cosh u}$ and that $\cosh u > \sinh u$.

Coth $u$ will approach unity as its limit as $u$ approaches infinity, decreasing as $u$ increases. This is immediately evident from the fact that the cotangent is the reciprocal of the tangent.

Sech $u$ approaches zero as $u$ approaches infinity decreasing from unity as $u$ increases. The hyperbolic cosecant also approaches 0 as its limit but decreases from infinity as $u$ increases.

Hence:- $\sinh \infty = \infty$, $\cosh \infty = \infty$, $\tanh \infty = 1$,
$\operatorname{sech} \infty = 0$, $\operatorname{cosech} \infty = 0$, $\coth \infty = 1$.

When $u$ varies from zero negatively the values of the hyperbolic cosine and secant are the same as for the positive values, while the values of the other functions are the negative. This follows directly from §6.

Hence we have the following conclusion:— cosh $u$ is never less than one, sinh $u$ can have all values, tanh $u$ varies from positive unity to negative unity, sech $u$ varies between zero and unity, cosech $u$ can have all values from positive infinity to negative infinity and coth $u$ values between infinity and unity and negative infinity and negative one.

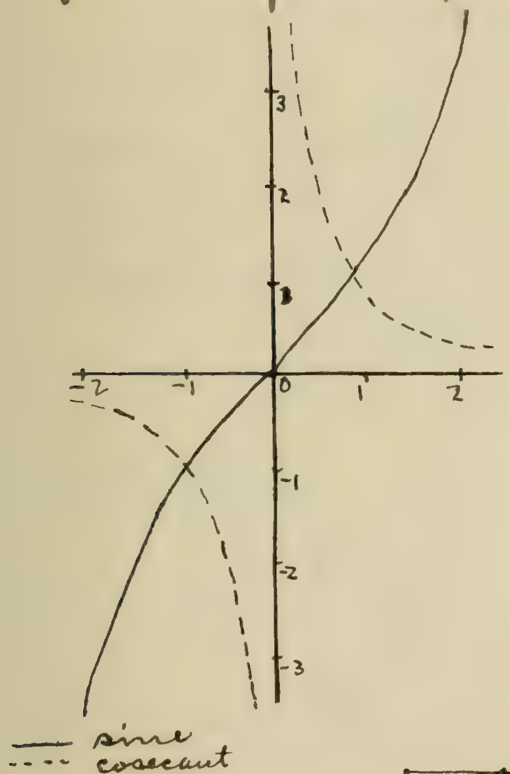
§16. Graphs of the Functions. By substituting for $u$ some numerical value in the series of §14, we obtain the corresponding values of the functions. By repeating this process, making $u$ vary at regular intervals, tables can be constructed, giving the values of the functions. Such tables have been worked out with great care by Gudermann[1] and numerous other investigators. Tables sufficient for ordinary use are given in McMahon's article on the Hyperbolic Functions[2]. The tables above referred

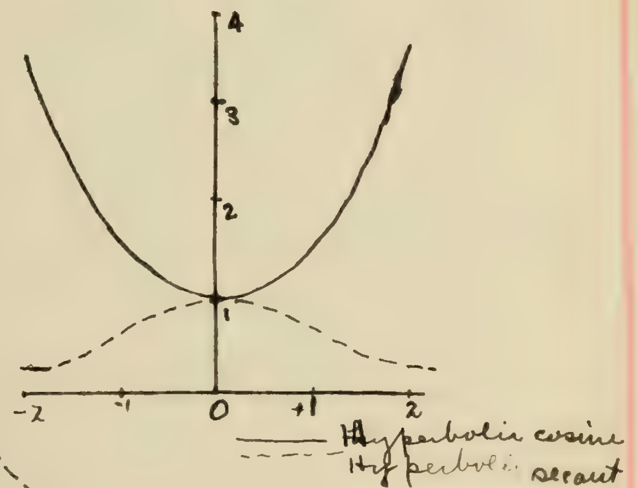
[1] See Journal der Mathemathik, vols. 7, 8, 9.

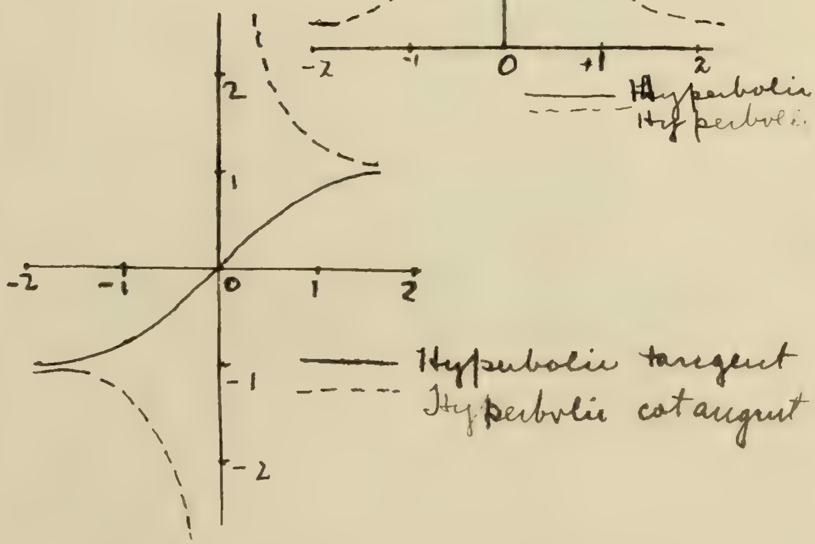
[2] See Merriman and Woodward Higher Math., pp. 162–8.

to, are accessible to most persons interested in the hyperbolic functions and are omitted here as being beyond the scope of the present thesis

By laying off values of $u$ as abscissae and the values of the functions as ordinates, curves representing the variation of the hyperbolic functions can be plotted. We give the curves for the different functions below:-




These curves illustrate clearly the conclusion of the preceeding section.

## §17. Series for the Inverse Hyperbolic Function

From §11 we have:–

$$\frac{d}{du}(\sinh^{-1}u) = \frac{1}{\sqrt{1+u^2}} = \frac{1}{(1+u^2)}^{-\frac{1}{2}} = (1+u^2)^{-\frac{1}{2}}$$

$$= 1 - \frac{1}{2}u^2 + \frac{1}{2}\cdot\frac{3}{4}u^4 - \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}u^6 + \dots$$

by the binomial theorem,

Integrating both sides of this equation,

$$\sinh^{-1}u = u - \frac{1}{2}\frac{u^3}{3} + \frac{1}{2}\cdot\frac{3}{4}\frac{u^5}{5} - \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\frac{u^7}{7} + \dots \quad (1)$$

the integration constant being zero since $\sinh^{-1}0 = 0$

This series is convergent only when $u$ is less than one and hence cannot be used for values greater than one. Another series convergent when $u > 1$ is obtained by writing $\frac{1}{u}(1+\frac{1}{u^2})^{-\frac{1}{2}}$ for $(1+u^2)^{-\frac{1}{2}}$ and expanding thus:

$$\frac{d}{du}(\sinh^{-1}u) = \frac{1}{u}(1+\frac{1}{u^2})^{-\frac{1}{2}}$$

$$= \frac{1}{u}(1 - \frac{1}{2}u^{-2} + \frac{1}{2}\cdot\frac{3}{4}u^{-4} - \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}u^{-6} + \dots)$$

Integrating

$$\sinh^{-1}u = C + \log u + \frac{1}{2}\cdot\frac{1}{2u^2} - \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{1}{4u^4} + \dots (2)$$

If $u = \infty$ we get

$$\underset{u \doteq \infty}{lt}(\sinh^{-1}u - \log u) = C$$

But by §5 $\sinh^{-1}u = \log(u+\sqrt{u^2+1})$

Hence $c = \underset{u \doteq \infty}{lt}\log\frac{u+\sqrt{u^2+1}}{u} = \underset{u \doteq \infty}{lt}\log(1+\sqrt{1+\frac{1}{u^2}})$

$= \log 2$

Hence $\sinh^{-1}u = \log 2u + \frac{1}{2}\cdot\frac{1}{2u^2} - \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{1}{4u^4} + \dots (2)$

when $u$ is greater than one

The $\cosh^{-1}u$ can be developed in a similar way

$$\frac{d}{du}(\cosh^{-1}u) = \frac{1}{\sqrt{u^2-1}} = (u^2-1)^{-\frac{1}{2}} = \frac{1}{u}(1-\frac{1}{u^2})^{-\frac{1}{2}}$$

$$= \frac{1}{u}(1 + \frac{1}{2}u^{-2} + \frac{1}{2}\cdot\frac{3}{4}u^{-4} + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}u^{-6} \dots)$$

$$\therefore \cosh^{-1}u = C + \log u - \frac{1}{2}\frac{1}{2u^2} - \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{1}{4u^4} - \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\frac{1}{6u^6} - \quad (3)$$

The value of $C$ is again found to be $\log 2$.

It is evident from § 15 that when $u$ is less than one, $\cosh^{-1}u$ is not real. Hence this series will not give real values when $u$ is less than unity, that is, it is not convergent.

Again $\frac{d}{du}(\tanh^{-1}u) = \frac{1}{1-u^2} = 1 + u^2 + u^4 + u^6 + u^8 \ldots$

Hence $\tanh^{-1}u = u + \frac{u^3}{3} + \frac{u^5}{5} + \frac{u^7}{7} + \frac{u^9}{9} + \ldots \quad (4)$

the integration constant being zero, since $\tanh^{-1}0 = 0$

$$\operatorname{sech}^{-1}u = \cosh^{-1}\frac{1}{u} = \log\frac{2}{u} - \frac{u^2}{2.2} - \frac{1.3\,u^4}{2.4.4} - \frac{1.3.5.\,u^6}{2.4.6.6} \cdots (5)$$

$$\operatorname{cosech}^{-1}u = \sinh^{-1}\frac{1}{u} = \frac{1}{u} - \frac{1}{2}\cdot\frac{1}{3u^3} + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{1}{5u^5} \cdots (6)$$

when $u$ is greater than unity,

and $$\operatorname{cosech}^{-1}u = \sinh^{-1}\frac{1}{u} = \log\frac{2}{u} + \frac{u^2}{2.2} - \frac{1.3u^4}{2.4.4} + \frac{1.3.5.u^6}{2.4.6.6} \cdots (7)$$

when $u$ is less than unity,

$$\coth^{-1}u = \tanh^{-1}\frac{1}{u} = \frac{1}{u} + \frac{1}{3u^3} + \frac{1}{5u^5} + \frac{1}{7u^7} + \cdots (8)$$

These series are convergent for all values of $u$ which make these functions real.

# Chapter V

## The Gudermanian

We have hitherto considered relations between the different hyperbolic functions and also the relations connecting the circular functions of imaginary argument with the hyperbolic functions of equal real arguments. The question naturally arises: "Is there any relation between the hyperbolic and circular functions when they are functions of real arguments only?" That there is will appear in the course of this chapter.

§18. The Gudermanian. If $u$ varies from zero to infinity, by §15, $\sinh u$ will vary from zero to infinity, taking any and all values between zero and infinity, provided $u$ be taken at intervals differing by sufficiently small quantities; in other words $\sinh u$ is a continuous function of $u$ as $u$ varies from zero to infinity. Again in the circular functions $\tan v$ varies from zero to infinity as $v$ varies from zero to $\pi/2$, taking any and all values between zero and infinity if $v$ is taken in sufficiently small intervals. Hence

we see that, if we find the value of $u$ corresponding to some value of $\sinh u$, it is possible to find a value of $v$ such that

$$\sinh u = \tan v,$$

since $\tan v$ varies continuously between zero and infinity as $v$ varies between zero and $\frac{\pi}{2}$, and $\sinh u$ varies from zero to infinity as $u$ varies from zero to infinity. Hence we see that for every value of $u$ between zero and infinity it is possible to find a corresponding value of $v$ between zero and $\pi/2$ and hence $v$ is a function of $u$ or the reverse. This correspondence or functional relation is expressed by saying that $v$ is the gudermanian of $u$ and $u$ is the anti gudermanian of $v$, written,

$$v = gd\, u \quad \text{and} \quad u = gd^{-1} v \quad \ldots\ldots (1)$$

If this relation connects $u$ and $v$, it is evident that

$$\sinh u = \tan v \quad \ldots\ldots (2)$$

Further $\cosh u = \sqrt{1+\sinh^2 u} = \sqrt{1+\tan^2 v} = \sec v$ (3)

Dividing (2) by (3)

$$\tanh u = \frac{\tan v}{\sec v} = \sin v \qquad (4)$$

Similarly $\operatorname{sech} u = \cos v$, (5)

$$\coth u = \operatorname{cosec} v \qquad (6)$$

$$\operatorname{cosech} u = \cot v \qquad (7)$$

These relations are easily remembered on account of their symmetry, viz.

$$\sinh u = \tan v, \qquad \tanh u = \sin v,$$
$$\cosh u = \sec v, \qquad \operatorname{sech} u = \cos v,$$
$$\coth u = \operatorname{cosec} v, \qquad \operatorname{cosech} u = \cot v.$$

Again by §1

$$\sinh u + \cosh u = e^u.$$

Hence
$$e^u = \tan v + \sec v$$
$$= \frac{1+\sin v}{\cos v}$$
$$= \frac{1-\cos(v+\frac{\pi}{2})}{\sin(v+\frac{\pi}{2})}$$
$$= \tan\left(\frac{v}{2}+\frac{\pi}{4}\right)$$

Therefore $\quad u = gd^{-1}v = \log_e \tan\left(\frac{v}{2}+\frac{\pi}{4}\right) \quad \ldots\ldots (8)$

It is apparent from this that if we had a table of natural logarithms of the tangents of angles, it would not be difficult to find the value of $u$ in terms of $v$. Tables of this kind have been constructed, but are not in general use. It would however be possible to find $u$ from a table of Briggsian logarithms by dividing by the modulus of the system.

Further $\tan\frac{v}{2} = \dfrac{\sin v}{1+\cos v} = \dfrac{\tan v}{1+\sec v}$

and $\qquad \tanh\frac{u}{2} = \dfrac{\sinh u}{1+\cosh u}$

Hence, since from above $\sinh u = \tan v$, and $\cosh u = \sec v$,

$$\tanh\frac{u}{2} = \tan\frac{v}{2} \quad \ldots\ldots\ldots (9)$$

At first glance it would seem that, since $\tan v = \sinh u$, $\tan\frac{v}{2} = \sinh\frac{u}{2}$ and therefore

$$\tanh\frac{u}{2} = \sinh\frac{u}{2},$$

which is impossible. The fallacy lies in the fact that $\qquad gd^{-1}\frac{v}{2} \neq \frac{1}{2}\,gd^{-1}v,$

which equality would have to be assumed in the above reasoning. However, the equation, to be correct, might be written

$$\tanh \tfrac{1}{2}\, gd^{-1} v = \sinh gd^{-1} \tfrac{v}{2}$$

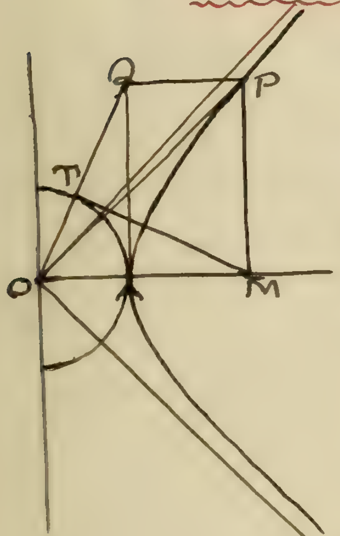
Similarly $\tan \tfrac{1}{2} gd u = \sin gd^{-1} \tfrac{u}{2}$,
giving easily a solution for

$$\tan x = \sin y$$

and

$$\tanh x = \sinh y.$$

§19. Geometric Proof. These formulae can all be proved geometrically as follows:




Let P be a point on a rectangular hyperbola. Let MT be a tangent to the circle of radius OA from the foot of the ordinate PM.

Further let $u = \dfrac{\text{sector } AOP}{OA^2}$ (see §3)

and $v = \angle AOT$.

Then $\sec v = \dfrac{OQ}{OA} = \dfrac{OM}{OA}$

and $\cosh u = \dfrac{OM}{OA}$.

Hence $\sec v = \cosh u$

which agrees with formula 3 of §18.

The above proof is for the case when $v$ is referred to the circle and $u$ referred to the rectangular hyperbola. They might just as well have been referred to the general ellipse and hyperbola, in which case $u$ and $v$ would simply be the ratios of the sectors to the triangles of

reference; but the ratio $v$ would be the same as the radian measure of the angle $v$ referred to the circle. To prove the formulas, use would have to be made of the theory of correspondence of points on ellipse and hyperbola. In the first case we treat of the gudermanian angle and in the second of the gudermanian function only.

§20 Differentiation of the Gudermanian Function. Let $v = gd\,u$.

Then $\tan v = \sinh u$.

Differentiating $\sec^2 v\, dv = \cosh u\, du$

But $\sec v = \cosh u$.

Hence $du = d\,gd^{-1}v = \sec v\, dv \quad \ldots (1)$

Again from (1) $dv = \cos v\, du$

$\qquad = \operatorname{sech} u\, du$.

Hence $d(gd\,u) = \operatorname{sech} u\, du \quad \ldots (2)$

We see therefore that the following integrals might be expressed:-

$$\int \sec v\, dv = gd^{-1}v \quad \ldots\ldots (3)$$

and

$$\int \operatorname{sech} u\, du = gd\,u \quad \ldots\ldots (4)$$

These agree with the usual forms of these integrals since

$$u = gd^{-1}v = \log\tan\left(\frac{v}{2}+\frac{\pi}{4}\right)$$

and

$$v = gd\,u = \tan^{-1}\sinh u\ *$$

---

* Cf. §12 formula 9

§21 Series for the Gudermannian and its Inverse. Graph. Since $d(gd u) = \operatorname{sech} u\, du$ and by §14, $\operatorname{sech} u = 1 - \frac{1}{2}u^2 + \frac{5}{24}u^4 - \frac{61}{720}u^6 + \dots$, by substituting and integrating, we obtain

$$gd\, u = u - \frac{1}{6}u^3 + \frac{1}{24}u^5 - \frac{61}{5040}u^7 + \dots \quad (1)$$

Similarly, by substituting in

$$d(gd^{-1} v) = \sec v\, dv$$

the series for $\sec v$ and integrating,

$$gd^{-1} v = v + \frac{1}{6}v^3 + \frac{1}{24}u^5 + \frac{61}{5040}u^7 + \dots \quad (2)$$

These series are however seldom used since there is no known law by which the coefficients progress. If it is necessary to find values, they are usually calculated from relations like

$$gd\, u = \tan^{-1} \sinh u$$

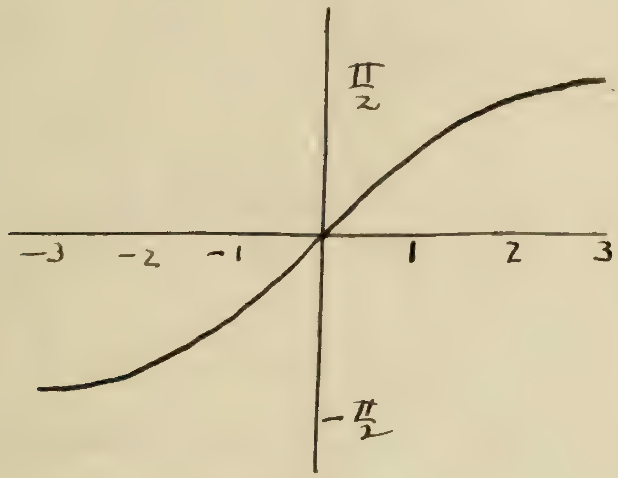
from tables of natural tangents and hyperbolic sines. Or use is made of formula 8 of §18,

$$u = \log_e \tan\left(\frac{v}{2} + \frac{\pi}{4}\right),$$

the use of which is discussed in that paragraph. For tables of $gd\, u$ and $gd^{-1} u$ the reader is referred to the excellent set given by Gudermann in vols 7, 8, 9, 10 of Journal der Mathematik.

As in the case of the hyperbolic functions, the gudermannian function can also be plotted, values of $u$ being taken as abscissas and the values of $gd\, u$ being taken as ordinates. The

curve is given in the figure below.




It is clear that the curve is asymptotic to $\frac{\pi}{2}$ and $-\frac{\pi}{2}$. Also that it is always less than $\frac{\pi}{2}$ and greater than $-\frac{\pi}{2}$.

# Chapter VI

## Hyperbolic Functions of Imaginary and Complex Arguments

Thus far we have treated only of the hyperbolic functions of real arguments. Imaginary and complex arguments have not yet been considered, and since no work on the hyperbolic functions is complete without a discussion of them, we shall close the consideration of the theory of these functions with a short discussion of these two kinds of arguments, before turning our attention to the applications of the hyperbolic functions.

§ 22. Pure Imaginary Arguments. We have shown in § 2 that the circular functions of imaginary arguments can be reduced to hyperbolic functions of equal real arguments. It would naturally seem possible to do the reverse, that is, express the hyperbolic functions of imaginaries in terms of circular functions of reals. For if in the formulae of § 2 we substitute $xi$ for $x$, we obtain

$$\cosh xi = \cos(x\,i\,i) = \cos(-x) = \cos x$$
$$i \sinh xi = i \sin(x\,i\,i) = i \sin(-x) = i \sin x$$
or $\sinh xi = i \sin x$.

Similarly, or by division,
$$\tanh xi = i \tan x$$
and so on for the other functions. The same results may be obtained by replacing $x$ by $xi$ in the definitions of §1 of the hyperbolic cosine and sine, but this really amounts to the same thing as the above.

It is apparent therefore that the hyperbolic cosine of a pure imaginary argument is real while the sine and tangent are imaginary.

§23. Complex functions. Since the addition formulae of §8 were proved independently of whether $u$ and $v$ were real or imaginary, they hold for all values of $u$ and $v$, and hence, for values of $u$ real, and $v$ imaginary. Therefore we have
$$\sinh(u+iv) = \sinh u \cosh iv + \cosh u \sinh iv.$$
But $\cosh vi = \cos v$ and $\sinh vi = i \sin v$.
Hence
$$\sinh(u+vi) = \sinh u \cos v + i \cosh u \sin v.$$
Also
$$\cosh(u+vi) = \cosh u \cosh vi + \sinh u \sinh vi$$
$$= \cosh u \cos v + i \sinh u \sin v,$$

When $vi$ is negative, the sign before the second half of the second member of each identity is changed to minus.

The above formulae show us that the hyperbolic sine and cosine of a complex argument are reducible to complex functions of the form $a+ib$, $c+id$. If

$$\sinh(u+vi) = a+ib$$

then
$$a = \sinh u \cos v$$
$$b = \cosh u \sin v,$$

and if
$$\cosh(u+vi) = c+id$$
$$c = \cosh u \cos v$$

and
$$d = \sinh u \sin v$$

This shows us that if we have a table of hyperbolic sines and cosine and also of the circular sines and cosine, we can easily calculate $a$, $b$, $c$, and $d$, and hence the hyperbolic sine and cosine of any complex number.

§24. Periodicity of the Hyperbolic Functions. If in the formulae of §23 $v$ takes the value $\frac{\pi}{2}$, we obtain:-

$$\sinh(u \pm \tfrac{\pi}{2} i) = \sinh u \cos\tfrac{\pi}{2} \pm i \cosh u \sin\tfrac{\pi}{2}$$
$$= \pm i \cosh u \quad \ldots\ldots (1)$$

and
$$\cosh(u \pm \tfrac{\pi}{2} i) = \cosh u \cos\tfrac{\pi}{2} \pm i \sinh u \sin\tfrac{\pi}{2}$$
$$= \pm i \sinh u \quad \ldots\ldots (2)$$

Again if $v$ is $\pi$, we obtain:-

$$\sinh(u \pm \pi i) = \sinh u \cos\pi \pm i \cosh u \sin\pi$$
$$= -\sinh u \quad \ldots\ldots \quad (3)$$
and
$$\cosh(u \pm \pi i) = \cosh u \cos\pi \pm i \sinh u \sin\pi$$
$$= -\cosh u \quad \ldots\ldots \quad (4)$$

If $v = \frac{3\pi}{2}$
$$\sinh(u \pm \tfrac{3\pi}{2} i) = \sinh u \cos\tfrac{3\pi}{2} \pm \cosh u \sin\tfrac{3\pi}{2}$$
$$= \mp \cosh u \quad \ldots \quad (5)$$
and
$$\cosh(u \pm \tfrac{3\pi}{2} i) = \mp \sinh u \quad \ldots \quad (6)$$

If $v$ is $2\pi$
$$\sinh(u \pm 2\pi i) = \sinh u \quad \ldots\ldots (7)$$
$$\cosh(u \pm 2\pi i) = \cosh u \quad \ldots\ldots (8)$$

and in general it is easily proved that if $n$ be any integer
$$\sinh(u \pm 2n\pi i) = \sinh u \quad \ldots (9)$$
$$\cosh(u \pm 2n\pi i) = \cosh u \quad \ldots (10)$$

This shows us that the hyperbolic functions, that is, the sine and cosine, are periodic functions the period being imaginary viz. $2\pi i$. This also appears from the relation between the circular and hyperbolic functions, since the period of the circular functions is $2\pi$, a real one. It is easily shown that
$$\tanh(u \pm n\pi i) = \tanh u$$
from which it appears that the tangent has the period $\pi i$.

§25. Let the $\cosh z = Z$ where $z$ is of the form $x + iy$ and $Z$ of the form $X + iY$. Further let the complex numbers $z$ and $Z$ be represented by

Argand diagrams in the usual way by points whose coordinates are $x, y, X, Y$.

If $z$ traces the line $x = m$ parallel to the $y$ axis, $Z$ will trace out an ellipse as is shown by eliminating $y$ from $X = \cosh m \sin y$ $Y = \sinh m$ ~~$\cos y$~~

since
$$\cos^2 x + \sin^2 y = 1$$
$$\frac{X^2}{\cosh^2 m} + \frac{Y^2}{\sinh^2 m} = 1$$

This will represent a series of confocal ellipses as $m$ varies since $\cosh^2 m - \sinh^2 m = 1$

Similarly if $z$ traces out a line parallel to the $x$-axis say $y = n$, $Z$ will trace out a hyperbola whose equation obtained by eliminating $x$ from the equations
$$X = \cosh x \sin n \quad Y = \sinh x \sin n$$
is
$$\frac{X^2}{\cos^2 n} - \frac{Y^2}{\sin^2 n} = 1$$

As $n$ varies this ~~also~~ represents a set of confocal hyperbolas also confocal with the ellipses above. The intersection of the ellipse and hyperbola represented by the above equation will give the value of $X$ + $Y$ for $x = m$, $y = n$ and hence the value for $\cosh(m + in)$ Hence if accurate diagrams of ellipses and hyperbolas be drawn for different values of $m$ and $n$ the value of $\cosh(m+in)$ can be read off at the intersection of the ellipse whose parameter is $m$ with the hyperbola of parameter $n$.

Similar diagrams can be constructed for the hyperbolic sine the equations being
$$\frac{X^2}{\sinh^2 m} + \frac{Y^2}{\cosh^2 m} = 1$$
and
$$\frac{X^2}{\sin^2 n} - \frac{Y^2}{\cos^2 n} = 1$$
a set of ellipses and hyperbolas the result turning the ellipses and hyperbolas of $\cosh(m+in)$ through a right angle, as is easily seen from the equations

# Chapter VII

## Applications of the Hyperbolic Functions to other Branches of Mathematics

No doubt the reader has before this point asked the question "What use can be made of these functions and the theory of them?" Like the circular functions, they enjoy a wide range of applications, and no doubt they deserve as wide an introduction into the mathematical sciences. The applications of these functions fall under two heads, viz.:- those to other branches of pure mathematics, which we shall treat in the present chapter, and those to applied mathematics, mechanics and physics, mostly through the medium of other mathematics.

§26. The first use of importance that can be made of these functions is to replace the longer expressions, like

$$\frac{e^x + e^{-x}}{2}, \quad \log(x + \sqrt{x^2 - a^2}) \quad \text{etc.},$$

by the shorter and simpler expressions:

$$\cosh x, \quad \cosh^{-1} x, \quad \text{etc.}$$

Since expressions of this nature often arise

in mathematics, it is much more convenient and expedient to use the notation of the hyperbolic functions corresponding to these expressions.

§27. Applications to Integral Calculus and Differential Equations. The most important application to the Integral Calculus arise from the analogy to the circular functions. By this analogy the remembering of many complicated formulae of the Integral Calculus is greatly simplified. For example, we have from Integral Calculus

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} = -\cos^{-1}\frac{x}{a}.$$

Further by § 13

$$\int \frac{dx}{\sqrt{a^2+x^2}} = \sinh^{-1}\frac{x}{a}$$

and $$\int \frac{dx}{\sqrt{x^2-a^2}} = \cosh^{-1}\frac{x}{a},$$

which carries us through all the fluctuations of signs possible between $x^2$ and $a^2$. Thus the student is led immediately to associate with the square root of the sum or difference of two squares, one being the square of the variable and the other a constant, in the denominator, the sine or cosine, the correct one being determined for the special case under consideration. The same is true for all other integrals involving inverse functions.

As to the differential equations we have already referred to one use in §11 in the solution of the problem: "What function of $x$ recurs in its second derivative?" as $\sinh x$ and $\cosh x$; so that the general solution of

$$\frac{d^2y}{dx^2} - y = 0$$

is

$$y = A\cosh x + B\sinh x.$$

Comparing this result with the solution of

$$\frac{d^2y}{dx^2} + y = 0,$$

as

$$y = A\cos x + B\sin x,$$

the connection is immediately apparent. Thus we might continue for many other similar equations.

§28. Application to the Theory of Equations.

With the aid of the hyperbolic functions it is possible to solve completely the general cubic equation. Every cubic equation can be reduced to the form

$$x^3 = bx + c \quad . \; . \; . \; . \; . \; . \; . \; . \; (1)$$

in which the sum of the roots $x + x' + x''$ is zero.

If we let $x = v\cosh u$,

$$v^3\cosh^3 u = bv\cosh u + c,$$

or

$$\cosh^3 u = \frac{b}{v^2}\cosh u + \frac{c}{v^3} \quad . \; . \; . \; . \; (2)$$

But by formula (8) of §8,

$$\cosh^3 u = \frac{3}{4}\cosh u + \frac{1}{4}\cosh 3u \; . \; . \; (3)$$

Equating coefficients we obtain:

$$\frac{b}{v^2} = \frac{3}{4} \qquad v = \sqrt{\frac{4}{3}b} \quad . \; . \; . \; . \; . \; (4)$$

and $\frac{c}{v^3} = \frac{1}{4}\cosh 3u$ ; $\frac{c\sqrt{3}}{2b^{3/2}} = \cosh 3u$ . . . (5)

This makes equation (2) an identity and hence $x = v\cosh u$ is a root of the equation. If we replace $3u$ by $\theta$, we obtain for the three roots of equation (1)

$$\left.\begin{aligned} x &= \sqrt{\tfrac{4}{3}b}\ \cosh \theta/3 \\ x' &= \sqrt{\tfrac{4}{3}a}\ \cosh(\theta/3 + \tfrac{2}{3}\pi\sqrt{-1}) \\ x'' &= \sqrt{\tfrac{4}{3}a}\ \cosh(\theta/3 + \tfrac{4}{3}\pi\sqrt{-1}) \end{aligned}\right\}(6),$$

in which the value of $\theta$ is obtained from equation (5). These three values of $x$ give us the three roots of the above cubic for all values of $b$ and $c$. However it is necessary to distinguish between several cases:-

I When $b$ and $c$ are positive, and $\frac{c\sqrt{3}}{2b^{3/2}}$ is greater than 1;

II When $b$ is positive and $c$ negative, but $\frac{c\sqrt{3}}{2b^{3/2}}$ still $> 1$;

III When $b$ is negative;

IV When $b$ is positive, but $\frac{c\sqrt{3}}{2b^{3/2}}$ is $< \pm 1$.

Case I. When $b$ and $c$ are positive and $\frac{c\sqrt{3}}{2b^{3/2}}$ is $> 1$.

In this case $\theta$ is real, and the values of the roots can be obtained from the above formulae. These can be reduced as follows:-

$$x = \sqrt{\tfrac{4}{3}b}\ \cosh \theta/3$$

$$\begin{aligned} x' &= \sqrt{\tfrac{4}{3}b}\ \cosh(\theta/3 + \tfrac{2}{3}\pi\sqrt{-1}) \\ &= \sqrt{\tfrac{4}{3}b}\left[\cosh\tfrac{\theta}{3}\cos\tfrac{2}{3}\pi + \sqrt{-1}\sinh\theta/3\sin\tfrac{2}{3}\pi\right] \\ &= \sqrt{\tfrac{4}{3}b}\left[-\tfrac{1}{2}\cosh\theta/3 + i\tfrac{\sqrt{3}}{2}\sinh\theta/3\right) \\ &= -\tfrac{x}{2} + i\sqrt{b}\ \sinh\theta/3 \end{aligned}$$

$$x'' = \sqrt{\tfrac{4}{3}b}\ \cosh\left(\tfrac{\theta}{3} + \tfrac{4}{3}\pi\sqrt{-1}\right)$$
$$= -\tfrac{x}{2} - i\sqrt{b}\ \sinh \theta/3$$

Case II. When $b$ is positive, $c$ is negative, but $\frac{c\sqrt{3}}{2b^{3/2}}$ is $>1$.
In this case $\theta$ is imaginary because $\cosh u$ is positive for all real values of $u$. However by 3-4,

$$\cosh(u + i\pi) = -\cosh u.$$

Hence
$$x = \sqrt{\tfrac{4}{3}b}\ \cosh\left(\tfrac{\theta}{3} + \tfrac{\pi}{3}\sqrt{-1}\right),$$
$$x' = \sqrt{\tfrac{4}{3}b}\ \cosh\left(\tfrac{\theta}{3} + \pi\sqrt{-1}\right) = \sqrt{\tfrac{4}{3}b}\ \cosh\tfrac{\theta}{3},$$
and
$$x'' = \sqrt{\tfrac{4}{3}b}\ \cosh\left(\tfrac{\theta}{3} + 5\tfrac{\pi}{3}\sqrt{-1}\right).$$

It is easily shown that
$$x = -\tfrac{x'}{2} + i\sqrt{b}\ \sinh \theta/3$$
$$x' = -\sqrt{\tfrac{4}{3}b}\ \cosh \theta/3$$
$$x'' = -\tfrac{x'}{2} - i\sqrt{b}\ \sinh \theta/3$$

Case III When $b$ is negative.
Since $\cosh(u + \tfrac{3}{2}\pi i) = \frac{\sinh u}{i}$, substitute $\theta + \frac{3}{2}\pi i$ for $\theta$ and we get:
$$x = -\sqrt{4/3\,b}\ \sinh \theta/3$$
$$x' = -\tfrac{x}{2} + i\sqrt{-b}\ \cosh \theta/3$$
$$x'' = -\tfrac{x}{2} - i\sqrt{-b}\ \cosh \theta/3$$

Case IV When $b$ is positive, but $\frac{c\sqrt{3}}{2b^{3/2}} < \pm 1$.
Substitute for $\theta$, $i\theta$ and pass to the circular functions. This gives
$$x = \sqrt{\tfrac{4}{3}b}\ \cos \theta/3$$
$$x' = \sqrt{\tfrac{4}{3}b}\ \cos\left(\tfrac{\theta}{3} + \tfrac{2}{3}\pi\right) = -\tfrac{x}{2} + \sqrt{b}\sin \theta/3$$
$$x'' = \sqrt{\tfrac{4}{3}b}\ \cos\left(\tfrac{\theta}{3} - \tfrac{2}{3}\pi\right) = -\tfrac{x}{2} - \sqrt{b}\sin \theta/3$$

This gives us the values of the roots of the cubic for all possible values of $b$ and $c$, and hence the above are the complete solution of the cubic

To use the formulae the only thing necessary is a set of tables of the hyperbolic sine and cosine.

§29 Some Problems the Solutions of which are simplified by the Introduction of the Hyperbolic Functions.

I To find the arc of the logarithmic spiral $y = a^x$.

The arc of any curve is given by

$$s = \int \sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy.$$

In this case $\frac{dx}{dy} = \frac{1}{a^x \log a} = \frac{1}{y \log a}$.

Let $\frac{1}{\log a} = M$ and $\frac{y}{M} = \sinh u$.

Then $dy = M \cosh u\, du$.

Hence $$s = M\int \sqrt{1+\frac{1}{\sinh^2 u}}\, \cosh u\, du$$
$$= M \int \frac{\cosh^2 u}{\sinh u}\, du,$$

and since $\cosh^2 u = 1 + \sinh^2 u$,

$$s = M\left[\int \frac{du}{\sinh u} + \int \sinh u\, du\right]$$
$$= M\left[\log \tanh \frac{u}{2} + \cosh u\right].$$

II To find the arc of the spiral of Archimedes:

$$r = a\theta.$$

The arc of any curve in polar coordinates is

$$s = \int \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\, d\theta.$$

Here $\frac{dr}{d\theta} = a$.

Hence $s = \int \sqrt{r^2+a^2}\, d\theta$.

Let $\theta = \sinh u$, $d\theta = \cosh u\, du$, $r = a\theta = a \sinh u$.

Then $$s = \int \sqrt{a^2 \sinh^2 u + a^2}\, \cosh u\, du$$
$$= a\int \cosh^2 u\, du,$$

or since $\cosh^2 u = \frac{1}{2}(\cosh 2u + 1)$,

$$s = \frac{a}{2}\int(\cosh 2u + 1)\,du$$
$$= \frac{a}{4}(\sinh 2u + 2u)$$
$$= \frac{a}{2}(\theta\sqrt{1+\theta^2} + \sinh^{-1}\theta).$$

III. To find the area of a zone of an oblate spheroid generated by revolving the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, or $\frac{x^2(1-\varepsilon^2)}{b^2} + \frac{y^2}{b^2} = 1$, about the $y$-axis.

In any surface of revolution the surface is
$$S = 2\pi\int_0^y x\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy.$$
Here $\frac{dx}{dy} = -\frac{y}{x(1-\varepsilon^2)}$.

Hence
$$S = 2\pi\int x\sqrt{1+\frac{y^2}{x^2(1-\varepsilon^2)^2}}\,dy$$
$$= \frac{2\pi}{1-\varepsilon^2}\int\sqrt{x^2(1-\varepsilon^2)^2+y^2}\,dy,$$
or since $x^2 = \frac{b^2-y^2}{1-\varepsilon^2}$,
$$S = \frac{2\pi}{1-\varepsilon^2}\int\sqrt{b^2(1-\varepsilon^2)+\varepsilon^2 y^2}\,dy.$$
Let $\frac{\varepsilon y}{b\sqrt{1-\varepsilon^2}} = \sinh u \quad dy = \frac{b}{\varepsilon}\sqrt{1-\varepsilon^2}\cosh u\,du.$

Then
$$S = \frac{2\pi b^2}{\varepsilon}\int\sqrt{1+\sinh^2 u}\,\cosh u\,du$$
$$= \frac{2\pi b^2}{\varepsilon}\int\cosh^2 u\,du$$
$$= \frac{\pi b^2}{2\varepsilon}(\sinh 2u + 2u)$$
$$= \frac{\pi b^2}{\varepsilon}\left(\frac{\varepsilon y}{b\sqrt{1-\varepsilon^2}}\sqrt{e^2y^2+b^2(1-\varepsilon^2)} + \sinh^{-1}\frac{\varepsilon y}{b\sqrt{1-\varepsilon^2}}\right)$$

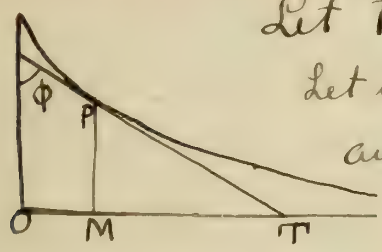
IV To find the radius of curvature of the hyperbola
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$
In general
$$R = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{d^2y/dx^2}.$$
Here $\frac{dy}{dx} = -\frac{b^2x}{a^2y}, \quad \frac{d^2y}{dx^2} = \frac{b^4}{a^2y^3}$

Then $$R = \frac{\left[1 + \frac{b^4x^2}{a^4y^2}\right]^{3/2}}{\frac{b^4}{a^2y^3}}.$$

Let $\frac{y}{b} = \sinh w$ and $\frac{x}{a} = \cosh w$.

Then $$R = \frac{(a^2\sinh^2 w + b^2\cosh^2 w)^{3/2}}{ab}.$$

V To find the equation of the curve, the length of whose tangent from the point of contact to the x-axis is constant




Let PT be the tangent at any point P. Let it make an angle $\phi$ with the y-axis and let its length be a. Then from the figure it is apparent that

$$y = a\cos\phi$$

Now $\phi = \tan^{-1}\frac{dx}{dy}$ from differential calculus.

Hence $$y = a\cos\tan^{-1}\frac{dx}{dy}.$$

Reversing we get

$$\frac{dx}{dy} = \tan\cos^{-1}\frac{y}{a} = \frac{\sqrt{a^2-y^2}}{y}$$

Integrating $$x = a\,\mathrm{sech}^{-1}\frac{y}{a} - \sqrt{a^2-y^2} + c$$

Let $y = a$ when $x = 0$. Then $c = 0$

Now $\frac{y}{a} = \cos\phi$.

Hence $$x = a\,\mathrm{sech}^{-1}\cos\phi - a\sin\phi = a(gd^{-1}\phi - \sin\phi).$$

This equation, together with $y = a\cos\phi$, gives a simple single-parameter form from which the curve can be easily found.

Let $OT = t$. Then $gd^{-1}\phi = \frac{t}{a}$.

If $t/a = w$, the above equations become

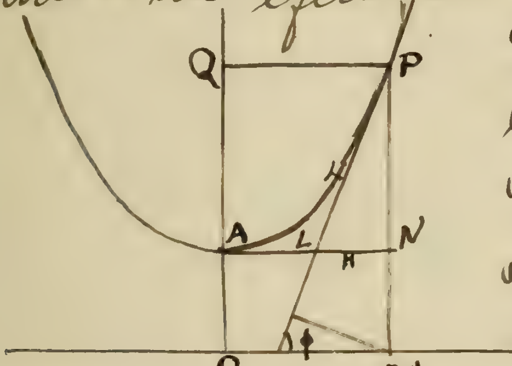
$$x = a\,gd\,w - \tanh w; \quad y = a\,\mathrm{sech}\,w.$$

This curve is commonly known as the tractory.

# Chapter VIII

## Applications of the Hyperbolic Functions involving Mechanical Problems.

§30. The Catenary. If a perfectly flexible uniform string be suspended by its ends what will be the equation of the curve so formed




Let $w$ be the weight of a unit length and let $s$ be the length of the portion $AP$. Then the weight of $AP$ is $ws$. Since the string is in equilibrium, the forces acting on it must be balanced. Other forces acting are $H$, the horizontal tension along the tangent at $A$, and $T$, the terminal tension along the tangent at $P$. Hence from $\triangle PLN$, where $PL = T$, $LN = H$, & $PN = ws$, if $\phi$ be the angle made by the tangent with the $x$-axis, it is evident that

$$T\cos\phi = H \quad \text{and} \quad T\sin\phi = ws.$$

Hence $$\tan\phi = \frac{ws}{H} = \frac{s}{a}, \text{ where } a = \frac{H}{w}.$$

But $\tan\phi = \frac{dy}{dx}$. $\therefore \frac{dy}{dx} = \frac{s}{a}$.

Then $$\frac{ds}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+s^2/a^2},$$

or $$\frac{ds}{\sqrt{s^2+a^2}} = \frac{dx}{a}.$$

Integrating, $\frac{x}{a} = \sinh^{-1}\frac{s}{a}$

and $\frac{s}{a} = \sinh x/a$ . . . . . . (1)

But $\frac{dy}{dx} = \frac{s}{a} = \sinh\frac{x}{a}$ .

Hence $\frac{y}{a} = \cosh\frac{x}{a}$ . . . . . . (2)

which is the required equation of the catenary.

From above, $\tan\phi = \sinh\frac{x}{a}$

Hence by §18, $x = a\,\mathrm{gd}^{-1}\phi$, and hence $y = a\sec\phi$ (3), the equation of the catenary expressed in terms of the angle which the tangent makes with the $x$ axis.

It is also easily shown that

$$y^2 = s^2 + a^2 \quad . \; . \; . \; . \; . \; (4)$$

The catenary is one of the most interesting curves in mathematics. Among its many properties might be mentioned, that the length of the perpendicular upon the tangent from the foot of the ordinate of the point of contact is constant and equal to the parameter $a$. Hence as seen from equation (4), viz.

$$y^2 = s^2 + a^2,$$

it is evident that the length of the tangent between the point of contact and the foot of this perpendicular is equal to the length of the curve from the lowest point A to the point of contact P. It is further evident that if a right triangle of which one leg is of constant length be moved so that the hypotenuse is always perpendicular to a given line and the variable leg equal to the length of

the curve traced by the point of intersection of this leg with the hypotenuse, the curve will be the catenary. It would seem not difficult to devise a mechanism fulfilling these conditions.

B. To find the center of gravity of arc AP. The arc of the catenary is by equation (1),

$$s = a \sinh \frac{x}{a} = a \sinh u \quad \left(u = \frac{x}{a}\right).$$

From Integral Calculus

$$\bar{x} = \frac{\int_0^x x\,ds}{\int_0^x ds} = \frac{a^2\int_0^u u \cosh u\,du}{s}.$$

Hence $$s\bar{x} = a^2\left[u \sinh u - \cosh u\right]_0^u$$

$$= a^2\left[u \sinh u - \cosh u + 1\right).$$

$$\bar{y} = \frac{\int_0^y y\,ds}{\int_0^y ds} = \frac{a^2\int_0^u \cosh^2 u\,du}{s}.$$

Hence $$s\bar{y} = \frac{a^2}{4}\left(\sinh 2u + 2u\right)_0^u$$

and $$4s\bar{y} = a^2\left(\sinh 2u + 2u\right).$$

C. To find the moment of Inertia of this arc about the terminal abscissa, QP.

In any case by calculus

$$I = \int r^2\,dm.$$

Here $r = (y_1 - y)$ $dm = \mu\,ds$, where $\mu$ = mass per unit length.

Hence $$I = \mu\int (y_1 - y)^2 ds$$

$$= \mu\left[\int y_1^2 ds - 2\int y_1 y\,ds + \int y^2 ds\right)$$

$$= \mu\left[y_1^2 s - 2y_1\bar{y}s + \int y^2 ds\right.$$

Now
$y = a\cosh u$ $\quad ds = a\cosh u\,du$

$$\therefore\ I = \mu\left[y_1^2 s - 2y_1\bar{y}s + a^3\int \cosh^3 u\,du\right]$$

$$= \mu\left[y_1^2 s - 2y_1\bar{y}s + a^3\left(\tfrac{1}{12}\sinh 3u + \tfrac{3}{4}\sinh u\right)\right).$$

Since $y_1 = a\cosh u$, $s = a\sinh u$; $s\bar{y} = \frac{a^2}{4}(\sinh 2u + 2u)$, we obtain

$$I = \mu[a^3\cosh^2 u\sinh u - a^3\cosh^2 u\sinh u - a^3 u\cosh u + \frac{a^3}{12}\sinh 3u + \frac{3a}{4}\sinh u]$$

$$= \mu a^3[\tfrac{1}{12}\sinh 3u + \tfrac{3}{4}\sinh u - u\cosh u].^*$$

§31. Center of Gravity and Moment of Inertia of the arc of the parabola $y^2 = 2px$.

Before finding the center of gravity, etc., it might be well to find the length of the arc with the help of the hyperbolic functions.

For any arc $\quad s = \int_0^y \sqrt{1 + (\frac{dx}{dy})^2}\, dy$.

Here $\frac{dx}{dy} = y/p$.

Therefore $\quad s = \int_0^y \sqrt{1 + y^2/p^2}\, dy$.

Let $y/p = \sinh u$. Then $dy = p\cosh u\, du$.

Hence
$$s = p\int_0^u \sqrt{1 + \sinh^2 u}\,\cosh u\, du$$
$$= p\int_0^u \cosh^2 u$$
$$= p/4(\sinh 2u + 2u)$$
$$= p/2\left(\frac{y}{p^2}\sqrt{y^2 + p^2} + \sinh^{-1}\frac{y}{p}\right).$$

The center of gravity is given by
$$\bar{x} = \frac{\int_0^x x\, ds}{\int_0^x ds} \quad \text{and} \quad \bar{y} = \frac{\int_0^y y\, ds}{\int_0^y ds}.$$

Hence
$$s\bar{x} = \int_0^x x\sqrt{1 + y^2/p^2}\, dy$$
$$= \frac{p^2}{2}\int_0^u \sinh^2 u\cosh^2 u\, du$$
$$= p^2/4\int_0^u \sinh 2u\, du$$
$$= \frac{p^2}{4}\int_0^u (\cosh 4u - 1),$$

* For a discussion of the catenary of uniform strength and the elastic catenary see Merriman and Woodward Higher Math. pp 147-9

or $s\bar{x} = \frac{p^2}{64}(\sinh 4u - 4u)$

and $64 s\bar{x} = p^2(\sinh 4u - 4u)$.

Similarly

$$s\bar{y} = \int_0^s y\,ds$$
$$= p^2\int_0^u \sinh u \cosh^2 u\, du$$
$$= \frac{p^2}{3}(\cosh^3 u)_0^u = \frac{p^2}{3}(\cosh^3 u - 1),$$

or $3s\bar{y} = \cosh^3 u - 1.$

The moment of Inertia of this arc about its terminal ordinates is

$$I = \int r^2 dm = \mu\int(x_1 - x)^2 ds$$
$$= \mu\left[x_1^2\int ds - 2x_1\int x\,ds + \int x^2 ds\right]$$
$$= \mu(x_1^2 s - 2x_1\bar{x}s + \frac{p^3}{4}\int \sinh^4 u \cosh^2 u\, du.$$

Now

$$\int \sinh^4 u \cosh^2 u = \frac{1}{8}\int(\cosh 2u + 1)(\cosh 2u - 1)^2$$
$$= \frac{1}{8}\int(\cosh^3 2u - \cosh^2 2u - \cosh 2u + 1)$$

$\cosh^3 2u = \frac{1}{4}\cosh 6u + \frac{3}{4}\cosh 2u$ and

$\cosh 2u = \frac{1}{2}\cosh 4u - \frac{1}{2}$.

Hence

$$\int \sinh^4 u \cosh^2 u = \frac{1}{8}\int\left[\frac{1}{4}\cosh 6u - \frac{1}{2}\cosh 4u - \frac{1}{4}\cosh 2u + \frac{1}{2}\right)$$
$$= \frac{1}{16}\left(\frac{1}{12}\sinh 6u - \frac{1}{4}\sinh 4u - \frac{1}{4}\sinh 2u + u\right).$$

Hence $I = \mu(xs(x - 2\bar{x}) + \frac{1}{64}p^3 N)$, where

$N = \frac{1}{12}\sinh 6u - \frac{1}{4}\sinh 4u - \frac{1}{4}\sinh 2u + u.$

§37. Applications to the Bending of Beams.

If a beam is built in at one end and a load P is applied at the other, and also a horizontal tensile force Q is applied at the same point, to find the equation of the curve assumed by the neutral

surface. If $(x, y)$ be any point on the surface with the free end as origin, the bending moment of this point is $Qy - Px$. Hence with the usual notation of the theory of flexure,

$$EI \frac{d^2y}{dx^2} = Qy - Px.$$

Let $\frac{Q}{EI} = n^2$ and $\frac{P}{EI} = mn^2$.

Then $\frac{d^2y}{dx^2} = n^2(y - mx)$.

If $y - mx = u$, $\frac{d^2y}{dx^2} = \frac{d^2u}{dx^2}$

and the equation assumes the form

$$\frac{d^2u}{dx^2} = n^2 u,$$

the solution of which is

$$u = A \cosh nx + B \sinh nx,$$

or $$y = A\cosh nx + B\sinh nx + mx.$$

The arbitrary constants $A$ and $B$ can be determined from the conditions of the problem. Since at the free end $x = 0$, $y = 0$, $A$ must be $0$, and hence

$$y = B \sinh nx + mx,$$

and $$\frac{dy}{dx} = nB\cosh nx + m.$$

At the fixed end $\frac{dy}{dx} = 0$ and $x = l$,

$$\therefore B = -\frac{m}{n\cosh nl}$$

Hence the required equation is

$$y = mx - \frac{m \sinh nx}{n \cosh nl}.$$

B. If the load is uniformly distributed over the beam $w$ per unit area, the equation assumes the form

$$EI \frac{d^2y}{dx^2} = Qy - \frac{1}{2} w x^2,$$

or, if $\frac{Q}{EI} = n^2$, and $\frac{1}{2}\frac{w}{EI} = n^2 m$,

$$\frac{d^2y}{dx^2} = n^2(y - mx^2)$$

Let $y - mx^2 = u$ and $\frac{d^2y}{dx^2} = \frac{d^2u}{dx^2} + 2m$.

Hence $\frac{d^2u}{dx^2} - n^2 u = -2m,$

the solution of which is

$$u = A\cosh nx + B\sinh nx + \frac{2m}{n^2},$$

or $$y = A\cosh nx + B\sinh nx + \frac{2m}{n^2} + mx^2.$$

Since at the origin $y = 0$, $x = 0$, $A = -\frac{2m}{n^2}$.

Hence $$y = \frac{2m}{n^2}\cosh nx + B\sinh nx + mx^2 + \frac{2m}{n^2},$$

$$\frac{dy}{dx} = \frac{2m}{n}\sinh nx + Bn\cosh nx + 2mx.$$

At the fixed end $\frac{dy}{dx} = 0$, $x = l$.

Hence $$B = \frac{-2ml - \frac{2m}{n}\sinh nl}{\cosh nl}.$$

Hence

$$y = \frac{2m}{n^2}\cosh nx - \frac{2mnl + 2m\sinh nl}{n\cosh nl}\sinh nx + mx^2 + \frac{2m}{n^2}$$

Many other problems of a nature similar to the above might be mentioned.

Hyperbolic functions are used also for alternating currents, which fall back on the solution of a differential equation of the form

$$\frac{d^2y}{dx^2} - a^2 y = F(x),$$

already discussed in §27, and hence we need not go into any detail here.

Applications to the fall of potential along a wire are found in Byerly's Fourier Series, pp 9 ff. In the same volume are given many other applications of the hyperbolic functions to problems in sound, heat and electricity.

## Historical.

The hyperbolic functions date their origin back to the eighteenth century. The foundations for these functions were laid long before their theory was actually developed. Unconsciously, Gregory a St. Vincentio (1584-1667) David Gregory (1667-1708) and Craig contributed to the early development by finding the area of the rectangular hyperbola. Newton further helped by drawing comparisons between the circle and the rectangular hyperbola, while De Moivre found that in the case of imaginary arguments the functions of the circle could be changed into similar real functions of the hyperbola. The first one who actually founded a theory of the hyperbolic functions was Vincenzo Riccati (1707-1775), who worked out the most important relations connected with them, from geometrical considerations. Lambert made use of this theory and extended it considerably, applying it in various ways, and especially to the solution of some trigonometric problems. For the next half century the advance made was comparatively slow. The next work of importance along this line was done by Gudermann and set forth in his article on the Potenzial Funktionen. The chief

addition which he made was the socalled gudermanian function*, connecting the hyperbolic and circular functions. He further worked out extensive tables for the hyperbolic functions and the gudermanian, and this is still one of the most complete set of tables of the functions published. Other investigators who have published tables of these functions are: Gronau (1860), Angelus Forti (1863), Hoüel (1866), Vassal (1872), and Ligowski (1873) and (1889).

Since the publication of Gudermann's article, the theory has been extended considerably, especially in the latter half of the nineteenth century. The chief extensions consist in various attempts at generalizing these functions. Perhaps the most interesting of these is set forth in Laisant's article "Essai sur les fonctions hyperboliques", (Paris, 1874), in which he defined the functions with respect to the general hyperbola instead of using the rectangular hyperbola, as had previously been done. In a similar way the circular functions have been extended to the ellipse. Another generalization was attempted by Günther (1881) by making them functions of the angle which

*This function was called by him the "longitudinal", expressed 'lu', and the inverse, the Längenzahl (Lu). The name gudermanian was first applied by Cayley: Elliptic Functions, 1876.

the radius vector of the curve $x^m - y^m = 1$ makes with the $x$-axis, the hyperbolic functions being the result of the special case in which $m = 2$. The corresponding extension of the circular functions is with reference to the curve $x^m + y^m = 1$, the circular functions being as above, the case where $m = 2$. A third generalization which might be mentioned was made by dropping out at regular intervals certain terms in the expansion

$$e^x = 1 + x + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \frac{x^4}{\lfloor 4} + \frac{x^5}{\lfloor 5} + \cdots,$$

thus

$$f_0(x) = 1 + \frac{x^n}{\lfloor n} + \frac{x^{2n}}{\lfloor 2n} + \frac{x^{3n}}{\lfloor 3n} + \cdots$$

$$f_1(x) = x + \frac{x^{n+1}}{\lfloor n+1} + \frac{x^{2n+1}}{\lfloor 2n+1} + \frac{x^{3n+1}}{\lfloor 3n+1} + \cdots$$

$$f_2(x) = \frac{x^2}{\lfloor 2} + \frac{x^{n+2}}{\lfloor n+2} + \frac{x^{2n+2}}{\lfloor 2n+2} + \frac{x^{3n+2}}{\lfloor 3n+2} + \cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$f_{n-1}(x) = \frac{x^{n-1}}{\lfloor n-1} + \frac{x^{2n-1}}{\lfloor 2n-1} + \frac{x^{3n-1}}{\lfloor 3n-1} + \frac{x^{4n-1}}{\lfloor 4n-1} + \cdots$$

The hyperbolic sine and cosine result when $n = 2$, for then we have

$$f_0(x) = 1 + \frac{x^2}{\lfloor 2} + \frac{x^4}{\lfloor 4} + \frac{x^6}{\lfloor 6} + \frac{x^8}{\lfloor 8} + \cdots$$

$$f_1(x) = x + \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} + \frac{x^7}{\lfloor 7} + \frac{x^9}{\lfloor 9} + \cdots,$$

which are clearly the series for the sine and cosine. These series had already suggested themselves to Riccati. They are discussed to a considerable extent in Glaisher "On a set of functions analogous to the circular functions"*. Günther has also treated them in his book on the hyperbolic functions.

* Quarterly Journal of Math., v. 16; 15-33.

An attempt in an altogether different direction was made by James Booth (1852) in the creation of a parabolic trigonometry, which is rather an extension of the circular trigonometry, the function being referred to the arc of a parabola. This trigonometry has not gained a very wide introduction into the mathematical sciences.

What the future will bring us in the work on these functions is difficult to say, but it is hoped that by degrees the importance of the hyperbolic trigonometry will be recognized, and that it will be placed on an equal footing with the now so common circular trigonometry.

## References.

1. Haskell; On the Introduction of the Notion of the Hyperbolic Functions. Bull. of Am. Math. Soc. v.

2. McMahon; Hyperbolic Functions. See Merriman and Woodward; Higher Mathematics pp. 109-68

3. Gudermann: Die Theorie der Potenzial Funktionen. See Crelle's Journal der Mathematik v. 6: 1-39; 162-94; 311-96.

4. Günther; Die Lehre von den gewöhnlichen und verallgemeinerten Hyperbelfunktionen, Halle, 1881.

5. Glaisher; On a set of functions analogous to the circular functions. Quart. Journ. of Math. v. 16.

6. Byerly: Fourier Series & Harmonic Functions, 15/8/

7. Rayleigh: Sound. v I art. 176; 477. 11/3

Good elementary discussions are found in

Loney's Trigonometry

Levett & Davison's Trigonometry.

For historical notes see

Braunmühl: Vorlesungen über Geschichte der Trigonometrie, vol. 2.